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THE FIXED POINT STATEMENTS ON BERGMAN TYPE COMPLEX MANIFOLDS

BY

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ABSTRACT

Under consideration are some nonclassical analogues of the Cartan fixed point theorem which are stated for meromorphic self-maps of complex manifold with nontrivial Bergman form. We discuss their exactness and examine possibilities for fixed point conditions on these manifolds.

1. Introduction

A noteworthy fact of complex analysis is that a holomorphic self-map $f\colon M \to M$ is an isomorphism under suitable assumptions on the complex manifold M and the mapping f. The basic result was obtained by Cartan in 1932 [1],[2]:

Let D be a bounded domain in $\mathbb{C}^n$ and let $f\colon D \to D$ be a holomorphic self-mapping of D with fixed point $x_0 \in D$. Then $|\det f'(x_0)| \leq 1$, and the equality holds iff $f \in \mathrm{Aut}(D)$. Moreover, a condition $f'(x_0) = \mathrm{Id}$ implies $f \equiv \mathrm{Id}$ identically.

During our discussion, it will be useful to replace simply the second statement in the following way.

Let D, f, x_0 be as above and

$$(*) \qquad f'(x_0) = \lambda \mathrm{Id}$$

with real $\lambda \geq 1$. Then $\lambda = 1$ and $f \equiv \mathrm{Id}$ identically.

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Lately, Kaup [7], Kobayashi (see review [8]), and Wu [12] have essentially extended the Cartan Theorem on the case of hyperbolic spaces (for some other developments based on a more detailed analysis of the Cartan phenomenon see, for instance, Graham [5],[6] and Kwack [9]). In our paper we state some analogs of Cartan's statements for meromorphic self-mappings $f \subset M \times M$ of the complex manifold M with nondegenerate Bergman form B_M (or, more specifically, with nondegenerate Bergman metric form b_M) in terms of the multiplicity function $N_f(x) = \# f^{-1}(x)$ and its upper bound $\sigma(f) = \text{ess}_M \sup N_f$. Our motivation is that, e.g., all the compact complex manifolds with trivial canonical bundle and any domain $D \subset \mathbb{C}^n$ of the finite Euclidean volume satisfy the condition $B_M > 0$ (and sometimes $b_M > 0$) everywhere, but they can be essentially nonhyperbolic. (See Rosay and Rudin's example in [10] and Example 2 below. In Example 2, after some modification of their arguments, we construct a holomorphic self-map $F: \mathbb{C}^n \to \mathbb{C}^n$ with $\det F'(z) \equiv 1$ whose image $F(\mathbb{C}^n) = Q$ has everywhere positive Bergman metric form b_Q.)

In such a situation Cartan's first statement transforms in the following way. Denote by $H(M)$ the space of all square-integrable holomorphic $(n,0)$-forms on M^n.

THEOREM 1: *Let M be a connected complex manifold and $B_M(x_0) > 0$ for some point $x_0 \in M$. If a meromorphic self-mapping $f \hookrightarrow M \times M$ keeps the point x_0 fixed, then the following statements hold:*

(1) $|\det f'(x_0)| \leq \sqrt{\sigma(f)}$,

(2) *the equality $|\det f'(x_0)| = \sqrt{\sigma(f)}$ holds iff $N_f(x) = \sigma(f)$ a.e. and there exists an element $w_0 \in H(M)$ such that $w_0(x_0) \neq 0$ and $f^* w_0 = \det f'(x_0) \times w_0$.*

COROLLARY 1: *Let $f \hookrightarrow M \times M$ be a bimeromorphic isomorphism of M which keeps some point $x_0 \in M$ fixed and $B_M(x_0) > 0$. Then $|\det f'(x_0)| = 1$ and there exists a form $w_0 \in H(M)$ which does not vanish at x_0 such that $f^* w_0 = \det f'(x_0) \times w_0$.*

COROLLARY 2: *Let M be a connected complex manifold of Bergman type (i.e. $B_M > 0$ everywhere). Then there is no bimeromorphic isomorphism $f \hookrightarrow M \times M$ which keeps two distinct points $x, y \in M$ fixed with the distinct Jacobians $\det f'(x) \neq \det f'(y)$.*

We show that Theorem 1 is (in some sense) exact, by Examples 1 and 2 below.

In order to generalize the second part of Cartan's Theorem we try to show that it is more or less natural to complete the assumptions $(*)$ above by the equality $|\det f'(x_0)| = \sqrt{\sigma(f)}$ together with a condition $b_M(x_0) > 0$. We obtain that $f \equiv \mathrm{Id}$ under such conditions in Theorem 2.

2. Background

Let M be a complex n-dimensional manifold and $H(M)$ denote the Hilbert space of holomorphic forms of bidegree $(n, 0)$ with the inner product

$$(\alpha, \beta) = \frac{(\sqrt{-1})^{n^2}}{2^n} \int_M \alpha \wedge \overline{\beta}.$$

The next proposition follows from [11].

PROPOSITION 1: *Let there exist $\nu \in H(M)$ for some point $x_0 \in M$ such that $\nu(x_0) \neq 0$. Then for any tangent vector $X \in T_{x_0}M$ all the functionals*

$$w \to \frac{w}{\nu}(x_0), \quad \omega \to X\left(\frac{w}{\nu}\right)(x_0)$$

are continuous on $H(M)$.

Let us assume that $H(M)$ is nontrivial and let $\{e_j\}$ be an orthonormal basis there. Then the Bergman form

$$B_M(x) = \frac{(\sqrt{-1})^{n^2}}{2^n} \sum_j e_j(x) \wedge \overline{e}_j(x)$$

is real analytic and does not depend on the choice of $\{e_j\}$ [11]. If $H(M) = 0$ then we write $B_M \equiv 0$ so that $B_M \equiv 0$ iff $H(M) = 0$.

Definition 1: We say M is a **Bergman-type manifold** if $B_M > 0$ everywhere.

(For instance, any complex torus T and any domain $D \subset \mathbb{C}^n$ with finite Euclidean volume $\mathrm{Vol}(D) < +\infty$ are of such type.)

Let us denote $E = \{x \in M \mid B_M(x) = 0\}$ and consider a local holomorphic coordinate system $z = (z^1, \ldots, z^n)$ near some point $x \in M \setminus E$. Then the Bergman metric form

$$b_M = \sum_{i,j=1}^{n} \partial_i \partial_{\overline{j}} \log |B_M / dz \wedge d\overline{z}| dz^i \otimes d\overline{z}^j$$

is semi-positive everywhere on $M \setminus E$. It is easily verified that $b_M(x) > 0$ iff $B_M(x) > 0$ and there exist n forms $w_1, \ldots, w_n \in H(M)$ such that $w_i(x) = 0$, $\partial_i\, w_i/dz(x) \neq 0$ for any $i = \overline{1, n}$.

Recall the famous Rosay–Rudin example of a strange holomorphic self-mapping $F: \mathbb{C}^n \to \mathbb{C}^n$ [10] which, after some refinement below, could be described by the following proposition.

PROPOSITION 2: *Let $n > 1$ and $z = (z^1, \ldots, z^n)$ be the usual holomorphic coordinate system on $\mathbb{C}^n$. Then for any $\gamma > 0$ there exists a holomorphic mapping $F = (F^1, \ldots, F^n)$ from $\mathbb{C}^n$ into $\mathbb{C}^n$, with $\det F'(x) \equiv 1$, so that*

$$\mathrm{Vol}_\tau(F^{-1}(\mathbb{C}^n \setminus \Delta^n)) < \gamma$$

where Δ^n is a unit polydisk and Vol_τ denotes a volume with respect to the form $\tau = (\sqrt{-1})^{n^2}/2^n \times \sum_{j=1}^n (|F^j|^2 + 1) dz \wedge d\bar{z}$.

Proof: We shall use a simple generalization of a lemma in [10, p.65].

LEMMA: *Let $n > 1$ and ν be an arbitrary continuous volume form on $\mathbb{C}^n$. Suppose that Q_0 and Q_1 are concentric open cubes in $\mathbb{C}^n$, with $Q_0 \subset Q_1$, and P is a cube in $\mathbb{C}^n$. Then there corresponds a holomorphic map $\phi: \mathbb{C}^n \to \mathbb{C}^n$ to every $\varepsilon > 0$ so that*

(a) $\det \phi'(z) \equiv 1$,
(b) $|z - \phi(z)| < \varepsilon$ *in* Q_0, *and*
(c) $\mathrm{Vol}_\nu\{z \in Q_1 \setminus Q_0: \phi(z) \notin P\} < \varepsilon$.

The original Rosay–Rudin statement in [10, p. 65] has been proved only for the Euclidean volume form $\nu = (\sqrt{-1})^{n^2} \times 2^{-n} dz \wedge d\bar{z}$ and our evident addition is that by a literal repetition of their arguments one can obtain this for our slightly general case.

The further proof is based also on [10] and although the Rosay and Rudin scheme is retained except for a few additional details, we repeat this in order to demonstrate their argument's potential and for the reader's convenience.

There are concentric open cubes Q_k in $\mathbb{C}^n$ so that $\overline{Q}_k \subset Q_{k+1}$ for $k = 0, 1, 2, \ldots, Q_1$ is our given cube Δ^n and $\mathbb{C}^n = Q_0 \cup Q_1 \cup \cdots$.

Assume, without loss of generality, that ε is so small that $w \in Q_1$ if $z \in Q_0$ and $|w - z| < \varepsilon$. Choose $\varepsilon_k > 0$ so that

(1) $\varepsilon_k < 1$, $k = 0, 1, 2, \ldots$,

(2) $\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \cdots < \varepsilon$,

(3) for any $j = 1, 2, \ldots$ the estimate $\varepsilon_{j+1} + \varepsilon_{j+2} + \cdots < \varepsilon_j$ holds.

Put $F_0(z) = z$. Assume, for some $k \geq 0$, that we have a holomorphic map $F_k\colon \mathbb{C}^n \to C^n$, with $\det F_k'(z) \equiv 1$, and that there is a cube $P_k \subset Q_k$ so that $F_k(P_k) \subset Q_0$ (this induction hypothesis holds when $k = 0$, with $P_0 = Q_0$). Choose $\delta_k > 0$ so that

$$(1) \qquad |F_k(z') - F_k(z'')| < \varepsilon_k, \quad \left||F_k^j(z')|^2 - |F_k^j(z'')|^2\right| < \varepsilon_k, \quad j = \overline{1, n},$$

whenever $z' \in Q_k$ and $|z'' - z'| < \delta_k$.

Let $\tau_k = (\sqrt{-1})^{n^2}/2^n \times \sum_{j=1}^n (|F_k^j|^2 + 1) dz \wedge d\bar{z}$. The lemma above furnishes a holomorphic map $\phi_k\colon \mathbb{C}^n \to \mathbb{C}^n$ and closed set $Y_k \subset Q_{k+1} \setminus Q_k$ with $\mathrm{Vol}_{\tau_k}(Y_k) < \varepsilon_k$, so that

(i) $\det \phi_k'(z) \equiv 1$,

(ii) $|z - \phi_k(z)| < \delta_k$ if $z \in Q_k$, and

(iii) $\phi_k((Q_{k+1} \setminus Q_k) \setminus Y_k) \subset P_k$.

Define $F_{k+1} = F_k \circ \phi_k$, and let P_{k+1} be some cube in $(Q_{k+1} \setminus Q_k) \setminus Y_k$. Then

$$F_{k+1}(P_{k+1}) \subset F_k(P_k) \subset Q_0.$$

This completes the induction step.

For all $z \in Q_k$ we have

$$(2) \qquad \begin{aligned} |F_{k+1}(z) - F_k(z)| &= |F_k(\phi_k(z)) - F_k(z)| < \varepsilon_k, \\ \left||F_{k+1}^j(z)|^2 - |F_k^j(z)|^2\right| &< \varepsilon_k, \quad j = \overline{1, n}, \end{aligned}$$

by (ii) and our choice of δ_k. Hence there exist

$$F = \lim_{k \to \infty} F_k, \quad \tau = \frac{(\sqrt{-1})^{n^2}}{2^n} \sum_{j=1}^n (|F^j|^2 + 1) dz \wedge d\bar{z} = \lim_{k \to \infty} \tau_k$$

uniformly on compact subsets of $\mathbb{C}^n$, $\det F'(z) \equiv 1$, and

$$(3) \qquad |F(z) - F_k(z)| \leq \sum_{\mu=k}^{\infty} |F_{\mu+1}(z) - F\mu(z)| \leq \sum_{\mu=k}^{\infty} \varepsilon_\mu < \varepsilon$$

for all $z \in Q_k$, by (2).

If $z \in Q_0$ then (3) implies that $|F(z) - z| < \varepsilon$; hence $F(z) \in Q_1 = \Delta^n$. If $z \in (Q_{k+1} \setminus Q_k) \setminus Y_k$ for some $k \geq 0$ then (iii) gives

$$F_{k+1}(z) = F_k(\phi_k(z)) \in F_k(P_k) \subset Q_0.$$

Also, $|F(z) - F_{k+1}(z)| < \varepsilon$, by another application of (3). As before, we conclude that $F(z) \in \Delta^n$. Any other $z \in \mathbb{C}^n$ lies in $\bigcup_1^\infty Y_k$. This completes the proof, because in view of the estimate $\tau_k \geq (\sqrt{-1})^{n^2} \times 2^{-n} dz \wedge d\overline{z}$ the Euclidean volume $\mathrm{Vol}(Y_k)$ is less than ε_k so that conditions (1)–(3), and relation (2), imply the next estimate for any $p \geq k \geq 1$:

$$\begin{aligned}\mathrm{Vol}_{\tau_p}(Y_1 \cup \cdots \cup Y_k) &\leq \sum_{\mu=1}^{k} \mathrm{Vol}_{\tau_p}(Y_\mu) \\ &\leq \sum_{\mu=1}^{k} \mathrm{Vol}_{\tau\mu}(Y_\mu) + n \sum_{\mu=1}^{k} \mathrm{Vol}(Y_\mu) \sum_{j=\mu+1}^{p} \varepsilon_j \\ &\leq \sum_{\mu=1.}^{k} \varepsilon_k + n \sum_{\mu=1}^{k} \varepsilon_\mu \varepsilon_\mu \leq (n+1)\varepsilon. \qquad \blacksquare\end{aligned}$$

Definition 2: A meromorphic mapping of the complex space M into the complex space N is the binary relation $f \subset M \times N$ such that there exists an analytic subset $S = S(f) \subset M$ with the following properties:

(1) $\mathrm{codim}_{\mathbb{C}}\, S(f) \geq 2$,
(2) a restriction $f|_{M \setminus S}$ is a holomorphic map, and
(3) the graph $\Gamma_f \subset M \times N$ is an analytic subset in $M \times N$.

If $f \subset M \times N$ is a meromorphic mapping then we denote

$$N_f(x) = \#f^{-1}(x), \quad x \in N,$$

the multiplicity function of f, and introduce (see [3] on some motivations)

$$\sigma(f) = \inf_A \sup_{x \in N} \#f^{-1}(x) \setminus A$$

where the infimum is taken over all analytic sets $A \subset N$ of dimension $\dim_{\mathbb{C}} A < \dim_{\mathbb{C}} N$ (by Remmert's theorem, an example of the situation $\sigma(f) < +\infty$ is a non-degenerate proper holomorphic map $f\colon M \to N$ for connected M, N). The proof of the next proposition may be easily obtained from [13], Lemma 2.12.

PROPOSITION 3: *Let $f \subset M \times N$ be a meromorphic mapping of the complex manifolds , with* $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} N = n > 0$. *If* $w, \nu \in H(N)$, *then*

$$\int_M f^* w \wedge \overline{f^* \nu} = \int_N N_f w \wedge \overline{\nu}.$$

3. Proofs

We start with the proof of Theorem 1. Due to trivial situations, let us suppose $\sigma(f) < +\infty$ and $a = \det f'(x_0) \neq 0$ (in view of Definition 2, the condition $f(x_0) = x_0$ implies immediately that f is holomorphic near x_0). By the condition $\operatorname{codim}_{\mathbb{C}} S(f) \geq 2$, together with Hartog's theorem, the form $f^* w$ is correctly defined on M whenever $w \in H(M)$. By Proposition 3 we have the following estimate:

$$\begin{aligned} \|f^* w\|^2 &= \frac{(\sqrt{-1})^{n^2}}{2^n} \int_M f^* w \wedge \overline{f^* w} \\ &= \frac{(\sqrt{-1})^{n^2}}{2^n} \int_M N_f w \wedge \overline{w} \leq \frac{(\sqrt{-1})^{n^2}}{2^n} \sigma(f) \int_M w \wedge \overline{w} \\ &= \sigma(f) \|w\|^2 \end{aligned} \tag{4}$$

so that the next linear operator

$$T_f(w) = a^{-1} \times f^* w \colon H(M) \to H(M) \tag{5}$$

has the norm

$$\|T_f\| \leq \sqrt{\sigma(f)}/|a|. \tag{6}$$

Since $B_M(x_0) \neq 0$, there exists a form $\nu \in H(M)$ such that $\nu(x_0) \neq 0$. Let us introduce the linear functional ℓ on $H(M)$

$$\ell(w) = \frac{w(x_0)}{\nu(x_0)}, \quad w \in H(M),$$

which is continuous on $H(M)$ by Proposition 1. According to (5) the operator T_f preserves ℓ, i.e.

$$\ell(T_f w) = \ell(w), \quad w \in H^2(M). \tag{7}$$

Let us examine the case when the estimate $|a| \geq \sqrt{\sigma(f)}$ holds. At first we shall show that such a situation is possible only if

(1) $|a| = \sqrt{\sigma(f)}$ and

(2) there exists a form $w_0 \in H(M)$ such that $w_0(x_0) \neq 0$, $f^*w_0 = aw_0$; moreover, $N_f \equiv \sigma(f)$ a.e.

Indeed, by our assumption $|a| \geq \sqrt{\sigma(f)}$ and (6), we have $\|T_f\| \leq 1$. Thus the operator T_f is the self-map of the closed unit ball $V_M = \{w \in H(M) \mid \|w\| \leq 1\}$:

$$T_f(V_M) \subset V_M. \tag{8}$$

For any $c \in \mathbb{C}$ let us consider the affine hyperplane $L_c = \{w \in H(M) \mid \ell(w) = c\}$ in $H(M)$. By (7) and (8) the intersection $D_c = L_c \cap V_M$ is preserved also under the T_f action:

$$T_f(D_c) \subset D_c.$$

Since ℓ is a continuous functional with respect to the norm of the space $H^2(M)$ and $\ell(\nu) = 1$, for sufficiently small $|c| < \varepsilon$ ($c \neq 0$), the intersection D_c will be a nonempty closed convex subset in V_M. Because $H(M)$ is a Hilbert space, D_c is a weak compact of $H^2(M)$ so that we may use the Schauder principle [4] for finding a fixed point $w_0 \in D_c$ of T_f:

$$T_f w_0 = w_0, \quad w_0(x_0) \neq 0;$$

hence

$$f^*w_0 = aw_0, \quad w_0 \in H(M), \quad w_0(x_0) \neq 0. \tag{9}$$

Let us integrate (9) over the manifold M:

$$\int_M f^*w_0 \wedge \overline{f^*w_0} = |a|^2 \int_M w_0 \wedge \overline{w_0}. \tag{10}$$

On the other hand, Proposition 3 allows us to write

$$\int_M f^*w_0 \wedge \overline{f^*w_0} = \int_M N_f w_0 \wedge \overline{w_0}. \tag{11}$$

Thus, both (10) and (11) imply

$$|a|^2 \int_M w_0 \wedge \overline{w_0} = \int_M N_f w_0 \wedge \overline{w_0}. \tag{12}$$

Since $w_0 \wedge \overline{w_0} \neq 0$ a.e. on M and for almost all $x \in M$ we have the estimate $|a|^2 \geq \sigma(f) \geq N_f(x)$, then (12) is true only if $|a|^2 = \sigma(f) = N_f(x)$ almost everywhere on M. This completes the proof of statement (1) and partially of statement (2) in Theorem 1. Now the theorem in complete form follows from equality (12), which allows us to state easily $|a| = \sqrt{\sigma(f)}$ under both conditions $f^*w_0 = \det f'(x_0) \times w_0$ and $N_f \overset{\text{a.e.}}{=} \sigma(f)$.

To prove Corollary 1 it is enough to apply Theorem 1 to both mappings f and f^{-1}. Corollary 2 follows from Corollary 1 and nonexistence of a pair of nonzero forms $w_1, w_2 \in H(M)$ with $f^*w_i = a_i w_i, \ \ a_1 \neq a_2$. ∎

Now, in order to examine the exactness of Theorem 1, we shall construct an example of the K-branched self-covering f of some Bergman-type manifold such that the equality $|\det f'(x_0)| = \sqrt{K}$ holds in fixed point x_0, with arbitrary large K.

Example 1: Let $M = T$ where T is a complex 1-dimensional torus. Thus $B_T > 0$ everywhere and there exists the isomorphism $\pi_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Choosing an arbitrary $t \in \mathbb{N}$ let us consider a subgroup $A \triangleleft \pi_1(T)$ generated by elements (tn, tp) where $n, p \in \mathbb{Z}$. Since such A has a finite index in $\pi_1(T)$ and we have $A \simeq \mathbb{Z} \oplus \mathbb{Z}$, then one can construct a K-branched holomorphic cover $g\colon T' \to T$ where T' is also some complex torus (and $K \geq t$). In terms of fundamental rectangles of both universal coverings $\mathbb{C} \to T$ and $\mathbb{C} \to T'$ it is easily verified that the moduli of T and T' coincide, hence we may write $T = T'$. Since T is a homogeneous complex manifold, a suitable composition $f = g \circ \varphi$ of g with some automorphism $\varphi \in \mathrm{Aut}(T)$ is a K-branched self-covering of T with at least one fixed point $x_0 \in T$.

Now we calculate the value of $|\det f'(x_0)|$ using the known fact $\dim_{\mathbb{C}} H(T) = 1$. Denoting by w_0 an everywhere nonzero holomorphic $(1,0)$-form on T we obtain $f^*w_0 = cw_0$, where $c = \det f'(x_0)$ by the equality $f^*w_0(x_0) = cw_0(x_0)$ in the fixed point x_0.

Integrating over T, we have

$$\int_T f^*w_0 \wedge \overline{f^*w_0} = |\det f'(x_0)|^2 \int_T w_0 \wedge \overline{w_0}.$$

On the other side, the relation $K = \sigma(f) = N_f(x)$ for all $x \in T$ together with Proposition 3 imply

$$\int_T f^*w_0 \wedge \overline{f^*w_0} = \sigma(f) \int_T w_0 \wedge \overline{w_0}$$

so that $|\det f'(x_0)| = \sqrt{\sigma(f)} = \sqrt{K}$.

The next example shows an error both of Cartan's Theorem and the statement $(*)$ for some complex domain with positive Bergman-metric form b_M.

Example 2: In accordance with Proposition 2, for any $n > 1$ there exists a holomorphic mapping $F = (F^1, \dots, F^n)\colon \mathbb{C}^n \to C^n$ such that $\det F'(z) \equiv 1$ and

$$(\sqrt{-1})^{n^2} \int_{F^{-1}(\mathbb{C}^n \setminus \Delta^n)} \sum_{j=1}^{n} (|F^j|^2 + 1) dz \wedge d\overline{z} < +\infty.$$

In particular, the image $Q = F(\mathbb{C}^n)$ is a domain and Proposition 3 implies

$$\begin{aligned}
&(\sqrt{-1})^{n^2} \int_{F^{-1}\mathbb{C}^n \setminus \Delta^n)} \sum_{j=1}^{n} (|F^j|^2 + 1) dz \wedge d\overline{z} \\
=&(\sqrt{-1})^{n^2} \int_{Q \setminus \Delta^n} N_f \sum_{j=1}^{n} (|z^j|^2 + 1) dz \wedge d\overline{z} \\
\geq&(\sqrt{-1})^{n^2} \int_{Q \setminus \Delta^n} \sum_{j=1}^{n} (|z^j|^2 + 1) dz \wedge d\overline{z}
\end{aligned}$$

so that

$$(\sqrt{-1})^{n^2} \int_{Q} \sum_{j=1}^{n} (|z^j|^2 + 1) dz \wedge d\overline{z} < +\infty.$$

Because all the functions $1, z^1, \dots, z^n$ are square-integrable over Q, then $b_Q > 0$ everywhere. Now we are in a position to construct a holomorphic self-mapping $f\colon Q \to Q$ which satisfies conditions $f(x_0) = x_0$, $f'(x_0) = A$ for some point $x_0 \in Q$ and an arbitrarily given $(n \times n)$-matrix A. Indeed, there exists a point $x_0 \in Q$ such that $F^{-1}(x_0) \cap Q \neq \emptyset$. Let us consider any point $x_1 \in F^{-1}(x_0) \cap Q$. Choosing the affine map $L\colon \mathbb{C}^n \to \mathbb{C}^n$, $L(x_0) = x_1$, $L'(x_0) = (F'(x_1))^{-1} \circ A$, we see that the restricted composition $f = F \circ L|_Q$ is a self-map of Q; moreover, $f(x_0) = x_0$ and $f'(x_0) = F'(x_1) \circ (F'(x_1))^{-1} \circ A = A$.

This allows us to examine the truth of the statement $(*)$ in the case of Bergman-type manifolds. Namely, as Example 2 shows, the assumption $[f'(x_0) = \lambda \mathrm{Id}, \lambda \geq 1]$ is not enough for this assertion even if $b_M(x_0) > 0$. Therefore, by Theorem 1 and Example 1 it is natural to use the equality $\det f'(x_0) = \sqrt{\sigma(f)}$ as an additional good condition in such a situation, as follows:

THEOREM 2: *Let $f \subset M \times M$ be a meromorphic mapping of the connected complex manifold M which keeps some point $x_0 \in M$ fixed and $b_M(x_0) > 0$. If $f'(x_0) = \lambda\mathrm{Id}$ with real $\lambda \geq 1$ and $\det f'(x_0) = \sqrt{\sigma(f)}$, then $f \equiv \mathrm{Id}$ identically.*

Proof: Since $\det f'(x_0) = \sqrt{\sigma(f)}$, then $N_f(x) \equiv \sigma(f)$ a.e. by Theorem 1, therefore the next equality holds by Proposition 3:

$$\begin{aligned}(f^*w, f^*\nu) &= \frac{(\sqrt{-1})^{n^2}}{2^n}\int_M f^*w \wedge \overline{f^*\nu} \\ &= \frac{(\sqrt{-1})^{n^2}}{2^n}\sigma(f)\int_M w \wedge \overline{\nu} = \sigma(f)(w,\nu), \quad w,\nu \in H(M)\end{aligned}$$

so that the operator $T_f = \sigma(f)^{1/2} \times f^*$ is an isometry of $H(M)$ into $H(M)$. Again by Theorem 1 there exists an element $w_0 \in H(M)$ such that $w_0(x_0) \neq 0$ and $T_f w_0 = w_0$. Let us define $E = \{w \in H(M) \mid w(x_0) = 0\}$ and observe that $T_f(E) \subset E$.

Choosing a holomorphic coordinate system $z = (z^1, \dots z^n)$ near x_0, we define n linear functionals on E

$$\ell_i(w) = \partial_i \frac{w}{w_0}(x_0), \quad w \in E$$

which are continuous by Proposition 1 (and are nontrivial by condition $b_M(x_0) > 0$). Since $f'(x_0) = \lambda\mathrm{Id}$, where $\lambda \geq 1$, then we have the following equality for any $w \in H(M)$:

$$\begin{aligned}\ell_i(T_f w) &= \partial_i T_f w/w_0(x_0) = \partial_i f^*w/\sqrt{\sigma(f)}w_0(x_0) \\ &= \partial_i f^*w/f^*w_0(x_0) = \partial_i w/w_0(f(x))\big|_{x=x_0} = \lambda\partial_i w/w_0(x_0) = \lambda\ell_i(w)\end{aligned}$$

which easily imply that $\lambda = 1$ and $\sigma(f) = 1$. Because operator T_f is a self-map of the weak compact $D_c = E \cap \{\|w\| \leq 1\} \cap \{\ell_i(w) = c\}$ whenever $c \in \mathbb{C}$, therefore, applying the Schauder principle for any $i = \overline{1,n}$, one may find n forms $w_1, \dots, w_n \in H^2(M)$ such that

$$w_i(x_0) = 0, \quad f^*w_i = w_i, \quad \partial_i w_i/w_0(x_0) \neq 0, \quad i = \overline{1,n}.$$

Evidently $\left\{\frac{w_i}{w_0}\right\}_{i=1}^n$ form a holomorphic coordinate system near x_0. Since $f \equiv \mathrm{I}$ in terms of $\left\{\frac{w_i}{w_0}\right\}_{i=1}^n$ and M is a connected manifold, we have the equality $f \equiv \mathrm{I}$ everywhere. ∎

In conclusion, note the following question (posed in this form after discussion with Professor Graham) which seems to be related to the statements above.

Let $f: M \to N$ be a holomorphic map of the connected Bergman Stein manifolds. If $\sigma(f) < \infty$ and $f^*b_N(x_0) = b_M(x_0)$ in some point $x_0 \in M$, is it true or not that $\sigma(f) = 1$?

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